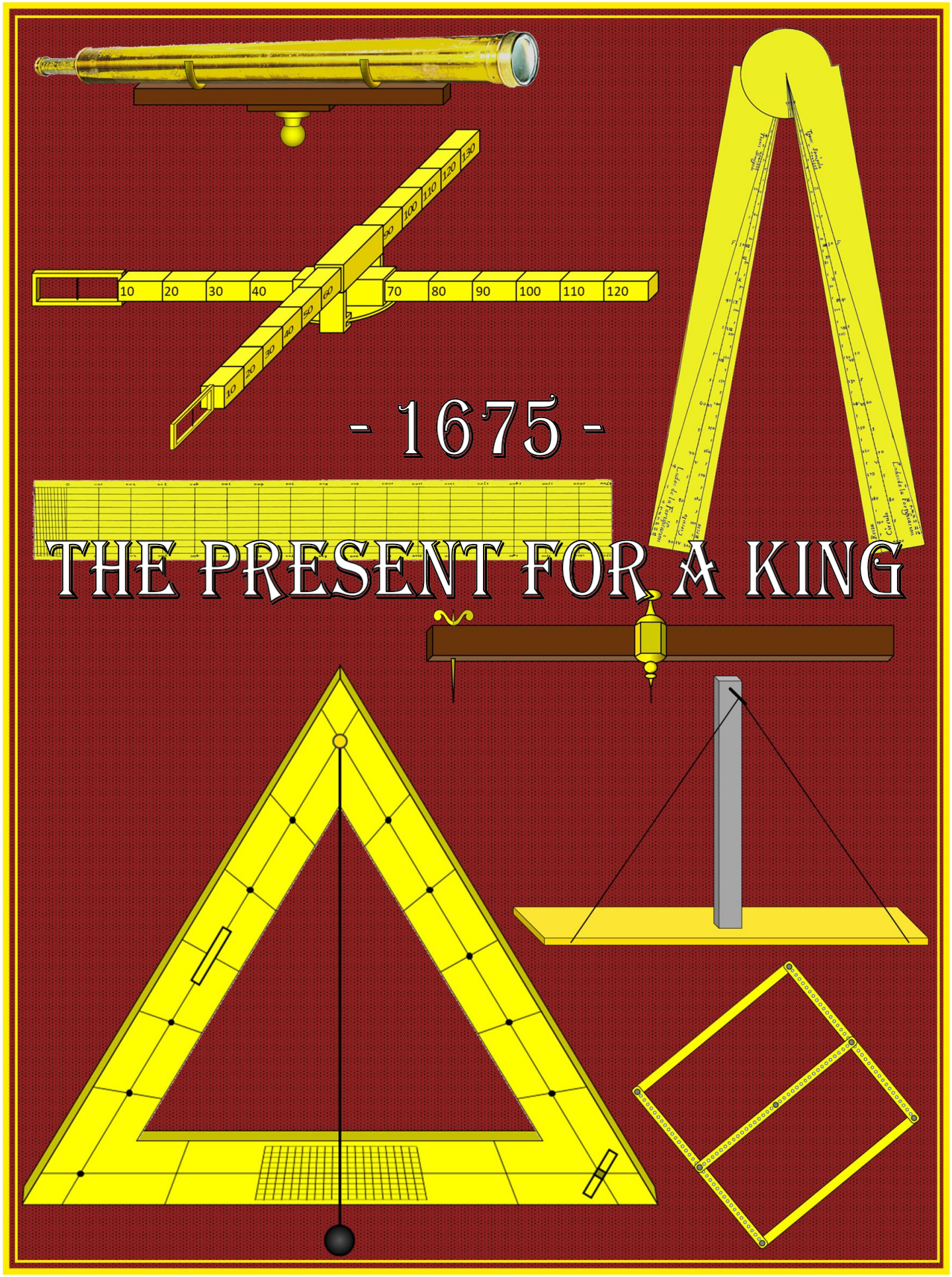

07-01-2023

By: Jose G. Fernández

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- 1675 – The Present for a King

Brief history

In 1675 Carlos II, at the age of 14, became king in his own right. One of the gifts he received was a chest with “14 mathematical objects”. Being 104.5 x 42 x 26 centimetres, it gives us a first impression of the size of the instruments it included, although, as we will see later, some were disassembled.

The gift had been prepared by an illustrious mathematician of the time: José Zaragoza, a 48-year-old Jesuit priest born in Castellón. Being also an astronomer and geographer, he held several teaching chairs and was related to other relevant figures in Europe at the time. This led him to be named teacher of the king and motivated the gift that we are referring to.

For us, collectors of slide rules and related instruments, it is worth noting the publication in 1672 of his book with tables of logarithms, the first in Spain.

An instruction manual is included in the chest, “Manufacturing and Use of Various Mathematical Instruments” [1], which describes the instruments, how to make them and their main uses. Most of them have a military and topographical application, with one musical exception, which is what initially caught my attention.

Documental References

My first contact with the instruments was through a study of José Zaragoza's mathematical-musical knowledge and its application in the so-called Harmonic Compass [2].

Although said study already includes a bibliography of the author, others can be found on the Internet (given the relevance of José Zaragoza), such as from the Royal Academy of History [3].

This last reference, and his authority on the subject, helps me to confirm the spelling of the surname, Zaragoza, since at the end of the book it appears as "Zaragozà" and it is also mentioned as such in [2].

After this introduction, just say that Zaragoza's book has been the guide for the development of this writing. In the Spanish version of this document, I have included some references to phrases in the book. Here they appear also translated.

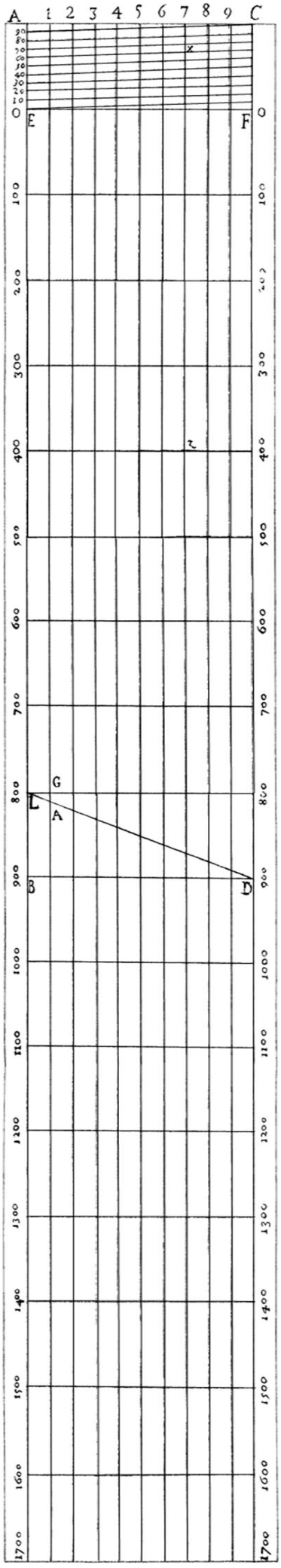
The present contents

When reviewing the list of the fourteen instruments, I think I would have not counted them in that way, but I am just including them below:

1- Brass ruler
2- Military “pantómetra” (first side of the sector)
3- Wired triangle
4- Geometric cross
5- Graphic rhombus
6- Equilateral triangle
7- Minor triangle
8- Telescope
9- Harmonic sector (second side of the sector)
10- Rod dividers, complement of the sector
11- Chain of 10 steps
12- Rosewood table, support or complement to any of the above instruments
13- Multipurpose stand, for the table and all the instruments.
14- Square ruler

To this list in the book, the author adds at the end "*a Flemish small box with various instruments*" (p.6) that I have not seen mentioned anywhere else in the book. This makes me think of a small box made of Flemish wood containing, perhaps, one or two pencils (at one point the author refers to the hardness of the pencil), a set of rulers (straight, square and to read angles), a pair of dividers and another with one end with a pencil, and maybe an ink block and a pen... In short, the accessories for working with maps, plans, drawings, etc.

I think that the fact that these instruments were a gift to a king indicates their importance for his time. And the detailed explanation of its possible applications has allowed me to get to know apparently simple instruments that allowed to solve technologically complicated issues at that time.



With some difficulty at first (the book is the original), but with great interest, I have been reading the chapters where the author explains the instruments. Here I include a summary for the main instruments, along with some mention of the secondary ones.

The Brass Ruler

In 1675 there was nothing like the international metric system we enjoy today, and things were measured in steps, “varas”, feet, cubits, hands, or fingers. Also, different regions used different values for example of what a foot measures. Father Zaragoza, after a description of this reality, tries to standardize his "standard" (the geometric foot) by choosing the value that seems most reasonable to him for the "ancient Roman foot" (there even were at least two values). From there, a finger is the sixteenth part of a foot, a hand is four fingers, the cubit is one foot and a half, and the step (geometric) is five feet.

As for the “vara”, today we would say that a “vara” is 2,768 geometric feet, but I think decimals were not used back then, so the author indicates that the relationship between a quarter of a “vara” and a foot is about 10,000 to 14,453. We will see that the proportion, or rule of three, is the basis for most of the operations in the book.

To give current values, in Wikipedia [4] we can read that the Castile “vara” is 83.6 cm, which means that the ancient Roman foot, or geometric foot, is 30.2 cm and the geometric step is 151 cm.

That being said, what does the ruler contain? The ruler is a brass plate one “vara” long (84 cm). One side is a table for the conversion of foot values according to cities, regions or countries, each showing the conversion factor in relation to one thousandth of the “Ancient Roman” foot. For example, if we divide the Roman foot into 1,000 parts, the Castilian foot would reach 923. The other side is a half “geometric step” rule, divided into 5,000 parts. Half a step is two and a half feet, so half a foot is divided by 1,000 (the distance AB in the figure, A at the top left, and B at the 900 line).

Anyone looking at the drawing will see that between A and B there are not the 1,000 divisions that we would expect from a ruler today. To begin with, the measurements were taken with dividers, and this supposes a different way of measuring. Imagine we want to open the compass at 467 “units” (ten thousandths of the geometric step). The 400 is marked by the distance between 0 and 400, since the "wide" divisions are only useful for the hundreds.

For the tens, we will look for the oblique line marked with "60" (top left) and then we will look for its intersection with the vertical line marked with "7", to have the units. Thus, we will place the points of the dividers between the crosses marked by the “z” and the “x” in the figure. In this way, it is only necessary to mark the units and the tens of the first hundred to be able to measure by thousandths.

In the figure, the ruler only reaches 1800 units, but that only indicates the limitation of the book, since the drawings are in true size. Thus, the page must measure about 27 centimetres; then, in real one “vara” (84 cm) ruler half a geometric step (75.5 cm) is included, with divisions reaching up to 5,000 (from 0 to 4,900 in hundreds, and the “extra” hundred with the divisions in tens and units).

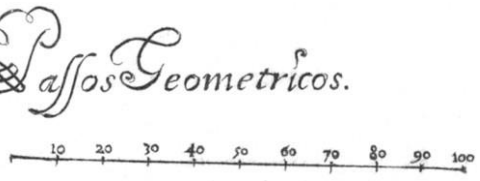
Regarding the “real size”, Father Zaragoza reveals that during the printing process the paper gets wet, which then shrinks when it dries. Although he recognizes that this shrinking of the wet paper is of the order of one sixtieth, he does not apply a correction so as not to distort the image of the real object, and thus, he only warns of the error.

<u>About generating drawings and maps</u>

Much of the book deals with generating maps or building drawings, or passing measurements from a drawing to a piece of land. Nowadays we clearly know what the scale of a drawing is (1:100,000...) but at that time this was not standardized either.

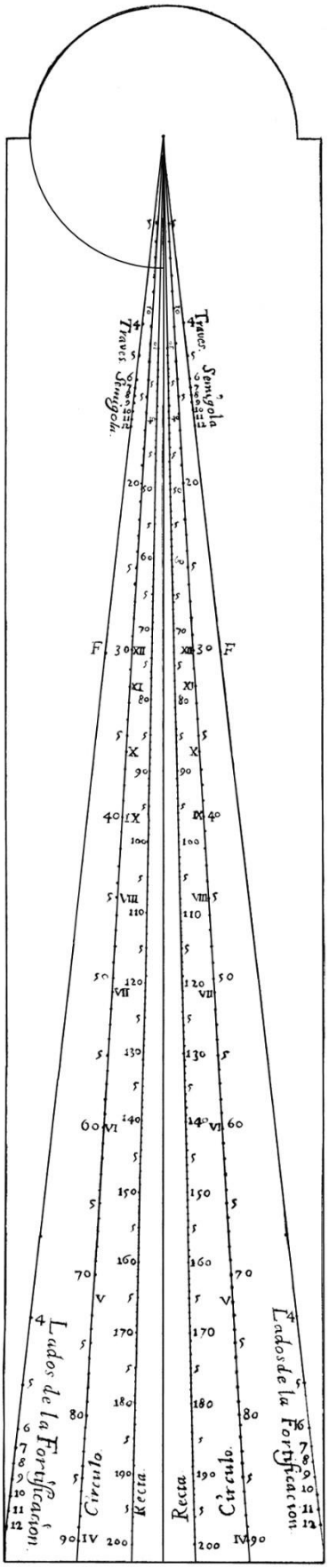
Due to this, the book speaks of the “pitipié” (from the French “petit pied”) and, given the units of the time, it is necessary to indicate if the scale is a “pitipié” of steps or feet, etc.

Drawing I, Figure 1



Drawing II, figure 7

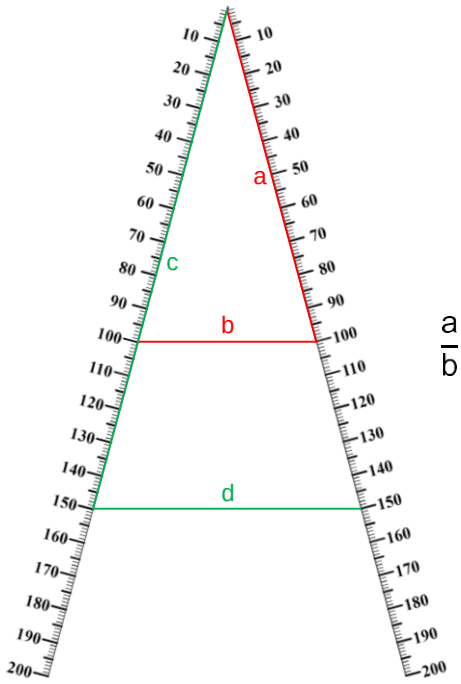
Military "pantómetra" (1st side of the sector)



The sector, or "pantómetra", is also one "vara" long (84 cm), and its scales are half a geometric step (75.5 cm). The first side is dedicated to calculation and building fortresses. And the second to music, which we will see in the next chapter.

The numbers scale

On this first face we have three double scales. The first, the numbers scale, from 0 to 500, (the picture only reaches 200) allows calculating with proportions (the rule of three), and, with them, multiplication and division. The operation is already well known [5] and based on the similarity of triangles:



$$\frac{a}{b} = \frac{c}{d} \Rightarrow a * d = c * b \Rightarrow \frac{a}{c} = \frac{b}{d}$$

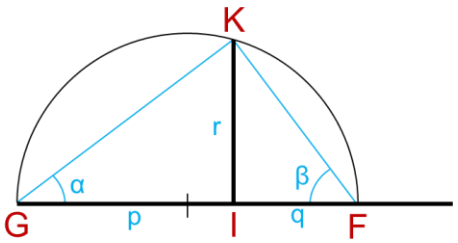
If we want to **multiply**, we can take "c" or "b" equal to 100 in the second equality. If we want to **divide**, we can take "d" equal to 100 in the first equality. Regarding **proportion**, for example, if we take the distance "100" from the scale of the drawing (at the "pitipié") with the dividers and open the sector so that the dividers fit as "b", then we can open the dividers to the distance "d", and it will be the value equivalent to "150" on the drawing. Or vice versa, taking a distance on the drawing and making it fit into the compass thus opened will give us the reading "c" of reality.

Furthermore, if at the beginning of a drawing we want to create the scale (the "pitipié") so that the "100" corresponds to a given distance "b", we will put that "b" among the "100" scales of the sector and so on. we will be able to transfer the 90, 80... to the new scale ("pitipié") just by passing the distances "90-90", "80-80"... from the sector to the drawing.

As for the **rule of three**, the author calls it the fourth proportional, alluding to knowing the fourth number when we know that "a" is to "b" as "c" is to "x". Another interesting calculation is when "b" and "c" are equal. This leads us to "$a * d = b^2$", so that "b" is the geometric mean (or proportional mean) between "a" and "d". Also, if we are able to separate a number in the product of two (for example if one is 10…), then we could find "b", its square root. But that would mean a trial-and-error process...

Drawing I, figure 2

To find “$b^2 = a * d$”, known “a” and “d” (such that “$a * d$” is also known), the author resorts to a graphical method. First, the sector is opened to an arbitrary value, and with the compass the distance between the values "a" and "a" of the scales is taken. On a drawn line, mark the distance just taken with the dividers, "GI". In the same way, take the distance from “d” to “d” on the sector with the dividers and add it to the line as distance “IF”. Considering now the distance "GF" on the line, get the midpoint and from this, draw the semicircle of diameter "GF". Then, from the previous point "I" raise a perpendicular to "GF", crossing the semicircle at point "K". Taking the distance "IK" to the sector with the dividers, it will be between "b" and "b" of the scales, being "b" the proportional mean between "a" and "d" or the square root of "$a * d$”.



Drawing II, figure 5

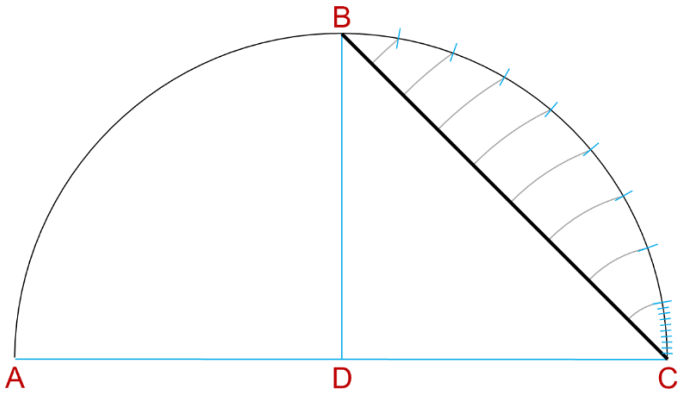
To understand the graphic calculation, I have added the lines in light blue. We have indirectly created two similar square triangles, where the sum of the angles α + β are 90º (the angle at the vertex K is square). In this way tan(α) = 1/tan(β), or , in similar terms:

$$\frac{r}{p} = \frac{q}{r} \Rightarrow r^2 = p * q$$

So, the sector is not "used" in this calculation, except to get some "GI" and "IF" values that fit on our paper, and to get the respective "IK" value at the end. You could take “a” and “d” directly from the sector scale (if these fitted the paper) and read “b” directly on it at the end.

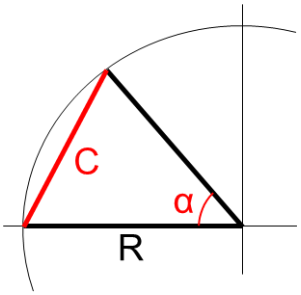
The scale of degrees

The second scale is for degrees, and goes from 0º to 90º. To create and draw it from degree to degree, José Zaragoza uses the division of a quarter of a circumference, with AB and BC being the length of these degree scales.



Drawing II, figure 4

These scales are actually of circumference chords, but the conversion of the chord “C” to the respective angle “α“ is easy, as long as we have the ratio between “C” and “R”, where the sector comes into play:



The radius "R" is equal to the chord "C" for an angle of 60º, so by putting "R" between "60" and "60" of the sector scales, we can find the respective chord for any other angle, just taking the distance between "α" and "α" on the scales; and vice versa, from the value of a chord and knowing the respective angle we can find the radius.

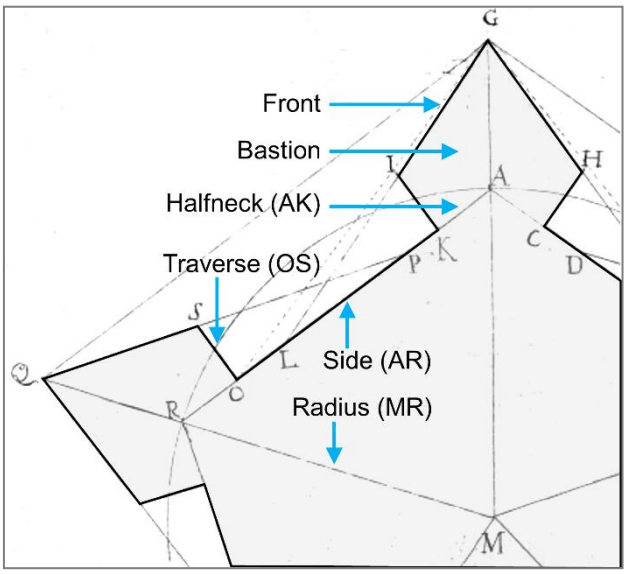
This has a direct application in the drawing of inscribed polygons, knowing the angle associated with each one. And this is marked on the scales in Roman numerals:

- Square: IV ➔ 90º
- Pentagon: V ➔ 72º
- Hexagon: VI ➔ 60º
- Octagon: VII ➔ 45º

And so on up to the dodecagon, (XII ➔ 30º), except for polygons with 7 and 11 sides, where it suggests trial-and-error or the ratio with respect to the side of a hexagon, such as 10,000 to 8,678 and to 5,635 respectively. I understand that it is because the angles of these two figures have decimals, so they cannot be obtained from the scale that only marks the degrees.

The scale for Fortresses

The ability to draw any polyhedral figure is the basis for this next section, since the plant of a fortress is based on regular geometric figures, or on parts of these if the terrain does not allow it. From what I understand from the book, at that time the fortresses were well defined and tabulated. Once the polygon that will be the base of the plant has been chosen, the proportions of the rest of the structures are taken from a table. Let's see a part of the plan of a pentagonal fortress, and its design elements:

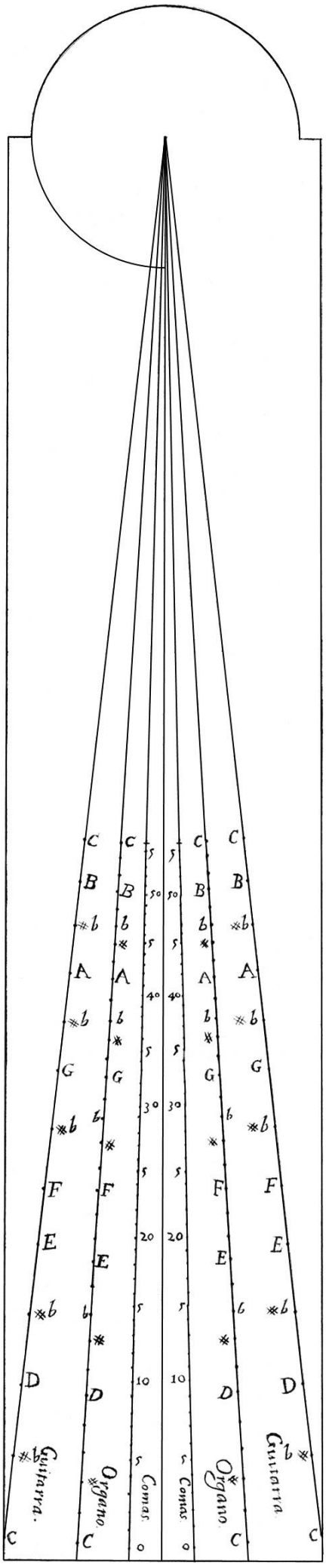


Drawing II, figure 7

The next step after drawing the polygon is to draw the bastions. To do this, you follow the data in a table:

Royal Fortresses Table					
Figure	**Radius**	**Side**	**Halfneck**	**Traverse**	**Angle**
(sides no.)	(steps)	(steps)	(steps)	(steps)	(degrees)
IV	103.1	146	23	23	90º 00’
V	131.0	154	27	27	72º 00’
VI	160.0	160	30	30	60º 00’
VII	185.2	162	31	30	51º 26’
VIII	214.1	164	32	30	45º 00’
IX	212.2	165	33	30	40º 00’
X	271.4	168	34	30	36º 00’
XI	301.3	170	35	30	32º 44’
XII	332.1	172	36	30	30º 00’
XV	418.2	174	37	30	24º 00’
XX	562.3	176	38	30	18º 00’
L	1414.1	178	39	30	7º 12’
Line	Infinite	180	40	30	Line
The front of the bastion is 64 steps in all figures					

Curiously, the radius column does include decimals, as well as the angle for polygons with 7 and 11 sides. After the bastions, the rest of the elements (moat, walkway, esplanade, walls...) also follow certain proportions, although they are more at the discretion of the author, and their values do not appear on the sector scale. This scale only includes data for figures with four to twelve sides.



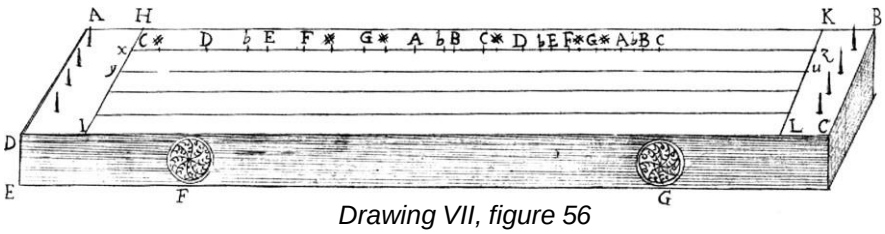
Drawing I, figure 3

Then, considering that the end of the scale corresponds to the 172 steps of the side of the 12-sided polygon, the distances of the other sides are included proportionally, labelling them with the number of sides of the respective figure (from 4 to 12). With the same proportion, the values of the halfneck are included, marking them with the same numbering. Just as the marks for the sides of the fortification are in the last third of the scale, the marks for the halfneck occupy the first third, making it easy to distinguish them. From the table, we also see that the traverse can be obtained from the halfneck. Finally, the proportional value of the length of the bastion front (64 steps) is in the central area and marked with the letter F.

We already have the angle on the degree scale, and the radius can be obtained on the sector from the angle and the side of the polygon in question. With this we have all the scales of this side of the proportion compass that José Zaragoza calls the Military Sector. Just add that, as we have seen, none of the scales is logarithmic.

The Harmonic Sector

On the back side of the sector, José Zaragoza included three scales for the manufacture of musical instruments (guitar, organ and clavichord), and even its tuning by means of another instrument, the tetrachord: a rectangular sound box with four equal strings on top. and equally tuned (unisons).



Drawing VII, figure 56

To explain how the scales are designed, the author develops a mathematical-musical treatise to define the "spaces" between notes, the harmonic intervals, according to harmonic consonance (if they sound good or not when played at the same time). In [2] there is a detailed evaluation of his explanations according to today's knowledge, while evaluating his conclusions and his relationship with other figures in history, from the time, precursors and successors.

The starting harmonic proportion is the musical octave (from one Do to the next Do, for example), such that the relationship between two notes one octave apart is 2 to 1. Today we can say that its frequency is double, but the simple explanation is that the relationship between the lengths of the strings that give each sound is double (or half).

Looking at the tetrachord image, for example. If the string at the back ("xz") is tuned to "C", by putting a bridge (to rest the string on) on its half, each half will sound like the next "C". I still have the doubt about how they did to tune the string to a "C" (tightening the string more or less with the pegs at each end, how did they know when the desired note was reached).

To study all musical concepts, my recommendation is the work of Klaus Kühn [6], since it presents the history of the musical process from a mathematical perspective (and with logarithms and slide rules).

As a summary I will say that there are some minimum proportions, the diesis (125 to 128) and the comma (80 to 81) from which, and by multiplication, one reaches the tones, the sixths, the fifths, the fourths, the thirds, and other musical intervals.

On the other hand, the ideal in an instrument is to achieve the "musical circle", where "*all the consonances can be found by going up or down* [keys or strings] *with the same proportion*". Although then he also says that achieving the circle is impossible, "*but removing the consonances from their place, so that they do not offend the ear, it is very easy, since it only consists of dividing the Octave or Diapason in equal parts*". (p.199)

The scale for the Organ

From there, for the manufacture of an instrument, you consider how many notes you want to include in an octave and how many parts (base proportions) will be included. For example, for the organ with five keyboards, the octave is divided into 31 parts (the tone is divided into five parts). Each part is equal to the previous one multiplied by the base proportion. Thus, if we have the lowest note (the string as is) divided into 10,000 units, the note one octave away will be in the middle of the string, at 5,000 units. So, mathematically, the base proportion is such that:

$$5.000 \cdot x^{31} = 10.000 \Rightarrow x = \sqrt[31]{\frac{10.000}{5.000}} \underset{[JZ]}{\Rightarrow} \log(x) = \frac{1}{31} \cdot \{\log(10.000) - \log(5.000)\} \Rightarrow x = 1,0226$$

As we can see, Zaragoza makes use of logarithms (from the tables that he had published three years earlier) to calculate the base proportion (he only explains the formula from "[JZ]", in p.200). And this leads to the table (the 5th in the book) for the organ scale (here the diesis is half a minor semitone):

C	5.000,00	s.2	7.150,56
b.1, s.2	5.113,05	s.1	7.312,24
b.2, s.1	5.228,67	F	7.477,58
B	5.346,89	b.1, s.2	7.646,66
b.1	5.467,79	b.2, s.1	7.819,57
b.2	5.591,43	E	7.996,38
s.2	5.717,86	b.1	8.177,19
s.1	5.847,15	b.2	8.362,09
A	5.979,36	s.2	8.551,16
b.1	6.114,56	s.1	8.744,52
b.2	6.252,82	D	8.942,24
s.2	6.394,21	b.1	9.144,44
s.1	6.538,79	b.2	9.351,21
G	6.686,64	s.2	9.562,65
b.1	6.837,84	s.1	9.778,88
b.2	6.992,45	C	10.000,00

If I'm not mistaken, in this scale the "C" corresponds to Do and the other notes follow the alphabetical order. The "s" is sharp and the "b" is flat. Regarding this division into 31 parts, it indicates the following:

"*This division is very old in music. Salinas mentions it, as something made in Italy, and is the author of more than a hundred years ago. N.Pomar, [Juan Bautista Pomar] a Valencian gentleman, made an organ with five keyboards that he presented to Your Majesty's father (who is already in heaven) and which was placed in the Royal Chapel… It was a prodigy that this gentleman without making calculations and with only the natural sense, came to achieve this division. Later, Félix Falcó de Belaochaga with Francisco Serrano, both well-versed in mathematics, searched for this division with numbers, although with great work for six months, because they did not have the formula that is explained in the previous paragraphs. Félix showed me the table and I gave him the rule with which he could have done it very easily*". (p.204)

The scale for the Guitar

For the Spanish guitar (although he says that it can also be applied to organs, clavichords and double-strung harps), he uses a division into 12 parts, or what is the same, use a base proportion such that:

$$5.000 \cdot x^{12} = 10.000 \Rightarrow x = \sqrt[12]{\frac{10.000}{5.000}} \Rightarrow \log(x) = \frac{1}{12} \cdot \{\log(10.000) - \log(5.000)\} \Rightarrow x = 1,0595$$

And this leads to the table (the 6th in the book) for the scale for the guitar:

C	5.000,00
B	5.297,31
s.b.	5.612,31
A	5.946,03
s.b.	6.299,65
G	6.674,19
s.b.	7.071,06
F	7.491,53
E	7.937,00

s.b.	8.408,97
D	8.908,99
s.b.	9.438,74
C	10.000,00

Both in this table and in the previous one I have put the numbers of the book. With a spreadsheet you can check the accuracy of them. Of this division into 12 parts, he comments:

"*This division, although it is the most common because it is that of the guitar, I think it is the least studied by musicians, since they have not used it assuming its imperfection, without demonstrating or experiencing it. The conveniences that it has are manifest in the guitar, because within its consonances it perfectly gives the Music Circle. And if it is applied to t he organ with the thirteen keys, which make twelve intervals, you will find everything that can be desired in music. And although it is true that it is with the imperfection that I have pondered before, the difference is not noticeable…*" (p.210)

And later he refers that having to renew the organ of the Royal Chapel, he asked the organ-builder to also make a small organ structured according to this configuration, for the use of King Carlos II.

The scale for the commas

This scale uses as base of proportion the comma (syntonic), that is, 80 to 81 (or 0.98765), and the table is:

0	10.000,000	29	6.974,998
1	9.876,543	30	6.888,887
2	9.754,610	31	6.803,839
3	9.634,183	32	6.719,841
4	9.515,243	33	6.636,880
5	9.397,771	34	6.554,943
6	9.281,749	35	6.474,018
7	9.167,159	36	6.394,092
8	9.053,984	37	6.315,152
9	8.942,207	38	6.237,187
10	8.831,809	39	6.160,185
11	8.722,774	40	6.084,133
12	8.615,086	41	6.009,021
13	8.508,727	42	5.934,835
14	8.403,681	43	5.861,566
15	8.299,932	44	5.789,201
16	8.197,463	45	5.717,729
17	8.096,260	46	5.647,140
18	7.996,306	47	5.577,422
19	7.897,586	48	5.508,565
20	7.800,085	49	5.440,558
21	7.703,788	50	5.373,391
22	7.608,680	51	5.307,052
23	7.514,745	52	5.241,533
24	7.421,971	53	5.176,823
25	7.330,341	54	5.112,912
26	7.239,843	55	5.049,789
27	7.150,462	56	4.987,446
28	7.062,185		

It stops at the value 56 because it already exceeds the octave, 5,000. He says of this scale that "*you can examine the commas that enter the octave and any of the other harmonic intervals.*" (p.217)

Marking the scales in the Harmonic Sector

Having already all the values, first you draw the three pairs of lines. Then the maximum distance is taken in the outer pair (for the Guitar) and it is passed radially to the other two (for the Organ and the Commas). In this way we have the point of 10,000 units of each scale (the first “C” both in “Guitar” and in “Organ” and the “0” in “Commas”). From there and keeping the proportion (using the "Numbers" scale on the other side of the Proportion Compass) all the other points will be marked. Naturally, the scales end in the middle of the scale line, once reached the octave. In the comma scale (which is a little over the half) you can also mark the value 5,000, of the octave, for example with a line to distinguish it from the scale divisions.

We have seen that in each scale the position of the next mark (musical note or comma) is equal to the position of the previous mark multiplied by a constant factor. In short, and as the author suggests, we can work with logarithms and for each mark we will add a constant to the previous one. But, in any case, we see that they are not logarithmic scales either.

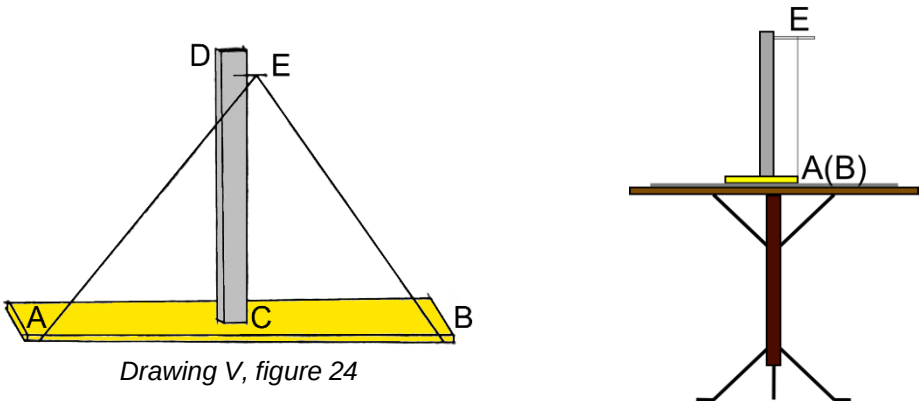
Use of the scales and of the tetrachord for tuning

The use of the scales is simple once you know the use of the scales on the other side of the sector. With a pair of rod dividers (see below) we take the measure of the string on the guitar (from the nut to the saddle or bridge, currently 65 cm) and, opening the sector, we place the dividers between the lower “C”. From there, the proportional distances will be taken from the sector and transferred to the guitar, marking the position of the frets.

It is done in the same way in the tetrachord, starting from the length between the “LK” and “HI” bridges, figure 56, and the points are marked according to the table or better with the sector. Its use, once the string is tuned to "C", consists of putting a mobile bridge on each of the marks and then playing the respective key or string of the instrument to be tuned, and tensing or loosening until both instruments sound the same (the strings are in unison). On the tetrachord a different scale can be marked under each string, and this will allow to tune various types of instruments.

The Wired Triangle

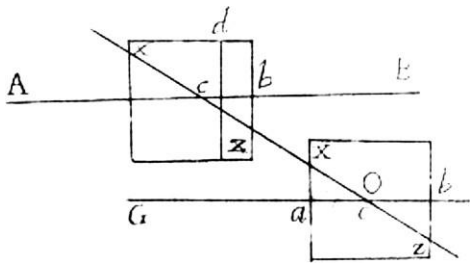
The wired triangle is an instrument for drawing lines on the ground. It is a vertical alignment sight, and it is complemented by a table having the drawing with the reference lines. On a "*ruler at least half a quarter wide and half a “vara” long*" (10.5 x 42 cm) "*made of wood or metal*" (p.95), "*another square ruler of one finger and perpendicular*" is set. (2 cm thick and I'm guessing about 21 cm tall), so that it is two fingers (4 cm) separated from one of the long edges of the base. A three-finger perpendicular nail (almost 6 cm) is fixed at the upper end of the vertical ruler, and a thread or wire is tied from the ends of the long edge of the base ruler to the nail, so that the triangle that describes the wire is perpendicular to the base (parallel to the vertical ruler).



Drawing V, figure 24

The plane containing AEB is vertical, and AB is the edge we use from the ground rule. By placing the instrument on a drawing on a table, we will be able to align AB with a line on the plane, and then looking from A, the vertical AE will allow us, for example, to align and nail a post with a string at the distance we want. On the other hand, by "placing" the table, drawing and instrument aligned on a line already drawn on the ground, we can turn only the instrument, so that AB is in line with another line on the drawing, at an angle with respect to the first. This will allow us to draw a new line on the ground (looking according to AE), which will keep said angle with respect to the initial line on the ground.

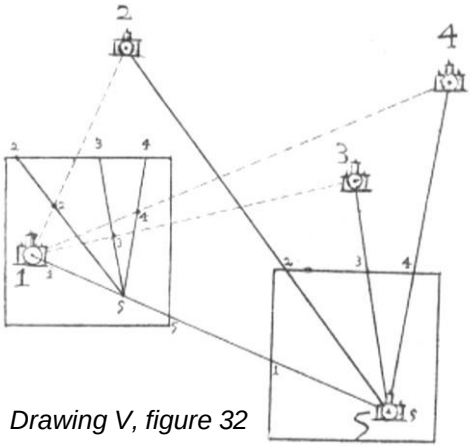
And, if we now take table, drawing, and instrument, and move them a certain distance along the newly drawn line, only returning the wired triangle to the position of the first line of the drawing, this will enable us to draw a third line on the ground and parallel to the first.



Drawing V, figure 26

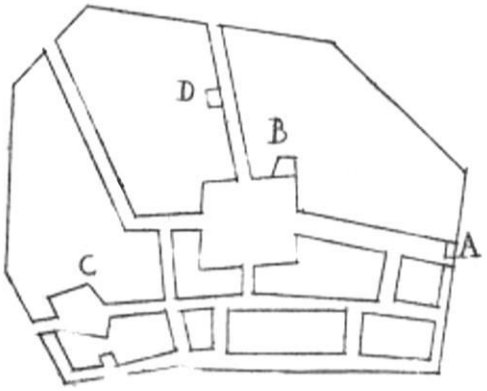
Another use alternative is mapping. For example, given points 1 to 5 in a region, from point 5 with the instrument on the table, we will draw the lines on a drawing as we see points 1 to 4 (and number them). Then, for example, we will move to point 1, we will align the table so that the line from 1 to 5 coincides and has the desired proportional distance (according to

a scale), and we will draw the lines as we now see the remaining points, 2, 3 and 4 (and we will number them). The intersections of the lines with the same number will place all the points on the map.



Drawing V, figure 32

The options are multiple. Father Zaragoza comes to recognize that he was able to trace the tunnels of a mine with this instrument, a light to mark the distant points at the end of each stretch of tunnel and something to measure the distances covered. The table with the instrument was used to mark the turning angles at each bend, intersection, etc.

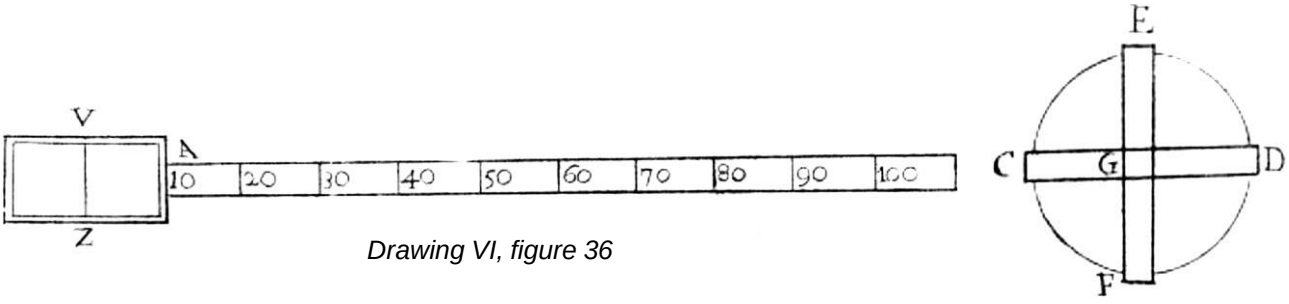


Drawing V, figure 35

The Geometric Cross

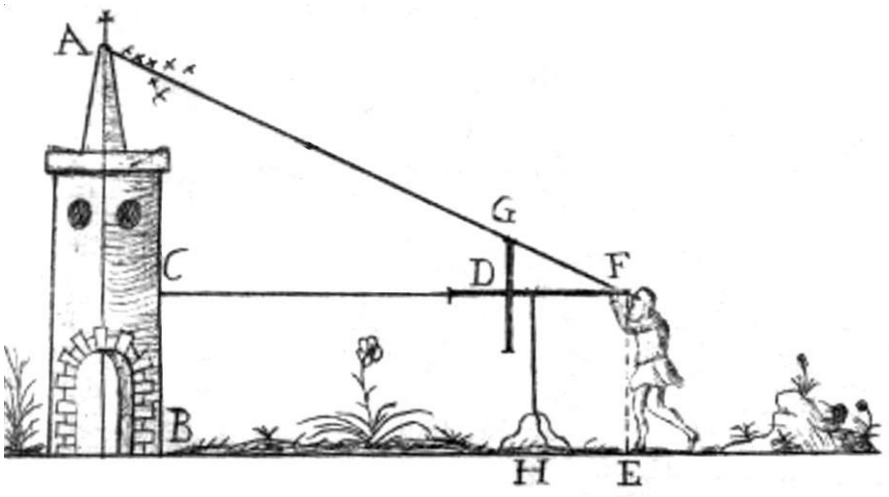
"*The geometric cross is an instrument that consists of two rods in the shape of a cross and is used to measure the terrain: heights, depths, and distances in a plain or in the mountains. Although it is called a cross, it does not always keep the arms at a right angle... No calculation is necessary, since it gives us the measure without the need to multiply or divide*" (p.121-122)

As Father Zaragoza explains it, it would look like a laser sight. And the truth is, tolerances aside, it does what he says. Two rods with a square section and one thicker than the other have marked divisions in the same unit (the value of the unit is not relevant). He recommends three to four feet in length (90 to 120 cm) for the thick stick, and the thin stick can be smaller. He also indicates some 3000 divisions in the thick rod and the corresponding ones in the fine one.



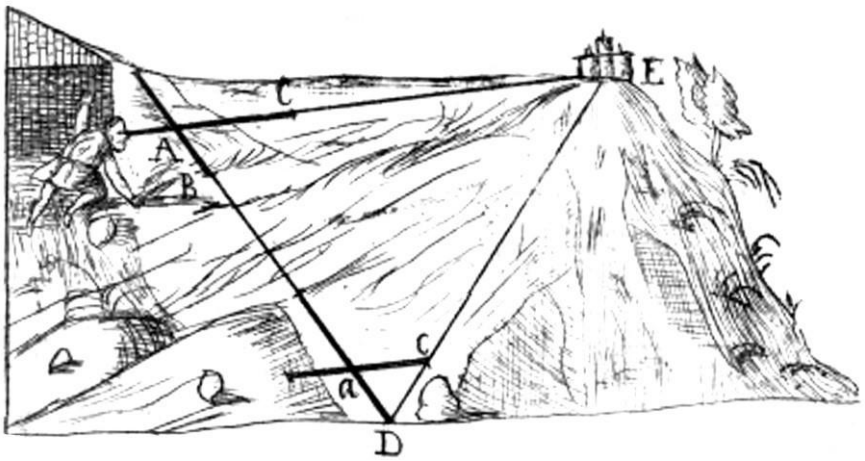
Drawing VI, figure 36

For fixing the rods in a cross, while allowing relative sliding, the Cross has in its centre a system of two square section tubes ("CD" and "EF") attached to a circular base. These tubes are 20 "units" long, and so you cannot see the central crossing point ("G") ±10 units (see reconstruction below). Finally, the instrument is attached to the multipurpose stand, by means of a ball joint that allows the cross to be placed horizontally (for horizontal distances) or vertically (for vertical distances). The use of the instrument is by similar triangles. For example:



Drawing VI, figure 37

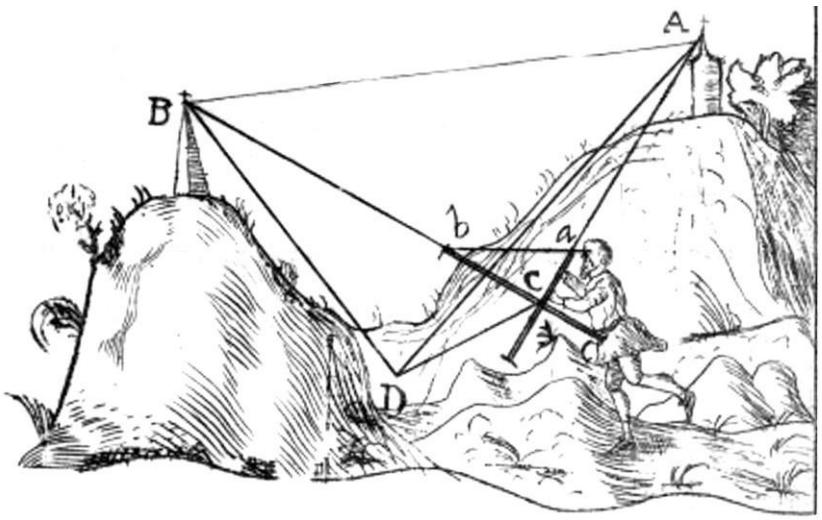
In the image, if we know the distance "BE", from the farthest point of the instrument to the tower, we can set that value in "units" on the larger arm (distance "DF"). If we then raise the smaller arm so that the end "G" is in line with "F" and "A", then the triangle "FAC" is similar to the triangle "FGD" and we can read the value of "CA" in the “units” from “D” to “G”. Finally adding the height of the stand, “DH”, we will have the height of the building, “AB”.



Drawing VI, figure 39

In this second example, we know the distance “AD” (we choose it). First of all, in position “A” we adjust the angle of the arms of the cross so that one arm points to the target “E” and the other to the known point “D”. We then adjust the length of the arm “AB” so that the “units” match the value of the known distance “AD”. We now take the cross to point D, and place it as shown, with the end called “B” (at point “A”) located at point “D”, (and being the adjusted distance “AB” now “aD ”). Then we move the other arm (without changing the angle between the arms) so that “C” is aligned with “D” and “E”. At this moment, the triangle "DaC" is similar to the "DAE", so that in the arm "aC" we can read the value of the distance "AE" in "units".

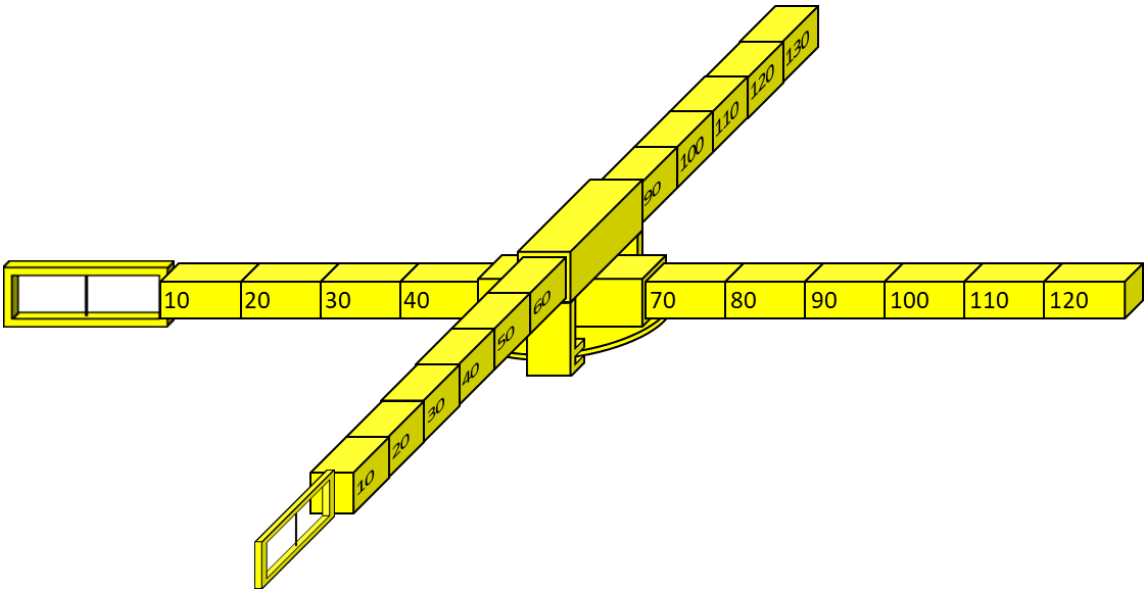
Father Zaragoza puts up to six examples (with their respective images) to show the great versatility of this instrument, ending at obtaining the distance "BA" knowing the distance "DC" in the following image, an exercise that I leave for the reader.



Drawing VI, figure 42

A recommendation given by the author is that "*one must ensure that the angles of the cross are not too acute"* because then the errors in the measurement "*always give little truth*" (p.133).

Finally, there is a paragraph that I could not understand, perhaps because I did not have the real instrument in my hands. In any case, I describe it here in case someone sees it clearly. I have prepared a recreation of what I think the instrument would be like, based on the author's images (there is a third one that I have not included here) and his explanations:



Each division in the drawn rods represents 10 units. So as for the sights (or pinnules), he says:

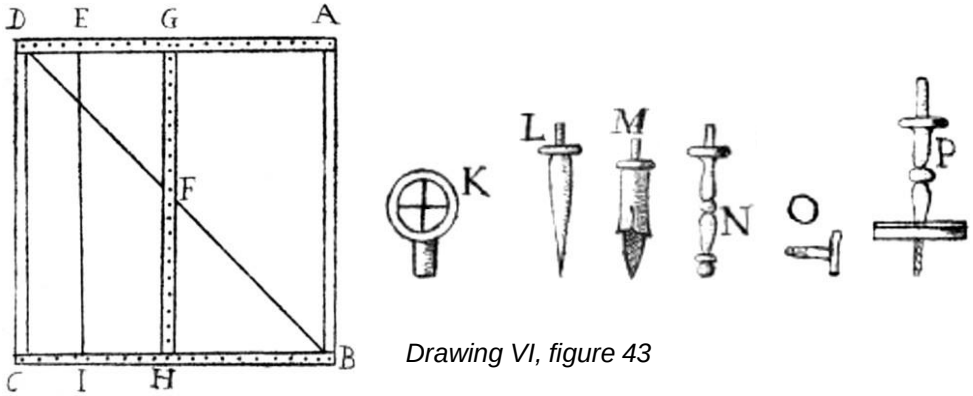
"*Two pinnules are necessary that are …20 rod divisions high. …they will be placed at the ends of the rods, at the first division. … On each rod… it has to be fixed on the head… so that the index hairline "VZ" is far from the point "A" as much as the division 10, and with this it will be equal to "GC". Since the 10 parts that "GC" covers are given by the sight. Then in all operations it is adjusted according to the number that the ends "C" or "E" cut, without the need to add or subtract*". (p.125)

As seen in my representation, and in figure 36 of the book (included above), the index hairline, "VZ", marks the "0" of the scale. So, to adjust the rod to the distance it must go from "0" on the thread to the centre of the cross (the "G" point in the original image), so I don't understand the "no need to add or subtract". For example, in my reconstruction the central point on the rod "parallel" to the paper is at "60", but in "C" we have "50" (in the oblique one the centre is at "80" and "E" at "70"). But he also says that "*the index hairline is as far from point "A" as division 10*", which in the image is already at "A", perhaps indicating either that the index hairline should be at "A", or that the first recorded number should be "20" (not "10"). In this way, the numbering would be displaced 10 units, and when reading at the end "C" or "E" of the central tubes, the value would correspond to that of the centre "G".

The Graphic Rhombus

"*The graphic rhombus is a universal instrument to copy a map, or the drawings of the fortresses, and also any image or painting, with the security of keeping the dimensions exactly proportional to the original. And it is equally easy to draw any building in perspective with respect to the point from which the person looks*" (p.135).

Again, a description of an instrument with great functionality. I already advance that it is the basic design of what today we call pantograph. The image of the instrument is:



Drawing VI, figure 43

It consists of five rods of equal length (for example, one foot, 30 cm) and fixed at the crossings without rigidity, allowing the rhombus deformation (hence its name). The author acknowledges that "CD" is not really necessary, or may be in another position, like "EI". “AD”, “BC” and “GH” have a series of holes at the same distances and numbered from “A” to “D”, from “B” to “C” and from “H” to “G”.

At the vertex "B" you place the pointer "L" (two fingers, 4 cm, long). At vertex “D” the pen for “*lead pencil*” “M”, “*which is soft and you can easily draw with it*” (p.137), with a total height like “D”. In "A" and "C" feet like "N", which may be slightly smaller than "L". “O” represents a simple screw (with an area without a thread so that once tightened the joint is still free to move), for points “G” and “H”.

Finally, the fixing foot "P" is placed in a point, “F”, between "G" and "H". In its lower part it has a thread to fix it to a table, and a square plate to ensure it stays perpendicular to said table. In its upper part another thread on a disk for fixing the rhombus. The distance between disk and square is like that of "N", so that once mounted, both "L" and "M" touch the papers well enough.

Horizontal copying (drawings, images …)

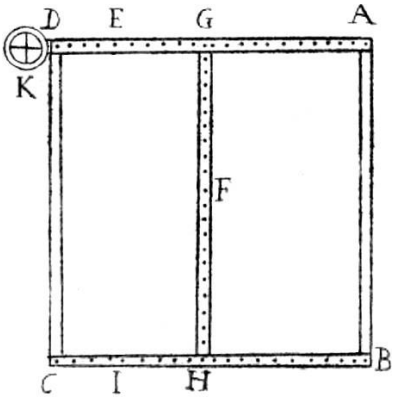
A 1 to 1 copy is made with the image settings and with the foot “P” in the centre of “GH”. If you want a copy larger than the original, place the rod “GH” closer to the end “B” than to “C” (and always perpendicular to “BC” and “AD”). The ratio of the original to the copy will be equal to the ratio of “BH” to “HC”. And vice versa if the copy is to be smaller than the original. The “P” foot on “GH” will be placed at the same distance “HP” (or “HF”, as the foot “P” is placed in “F”) as “BH”. As a check, points "B", "F" and "D" must always be aligned.

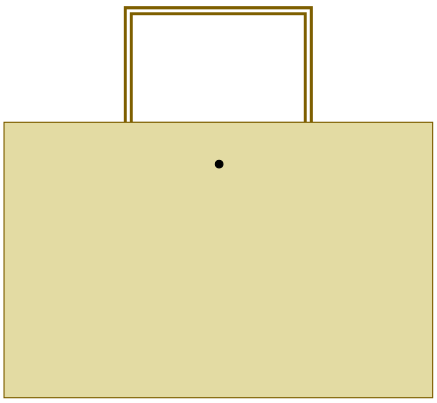
For example, we put “GH” in holes “13” of both “AD” and “BC” (position 13, 13). And if the rods have 51 holes (50 spaces) then the ratio will be 12 spaces (of the original) against 38 (of the copy). From here, proceed as with a modern pantograph: with the “L” pointer you follow (loosely) the entire original while the “M” pencil draws the copy (but inverted).

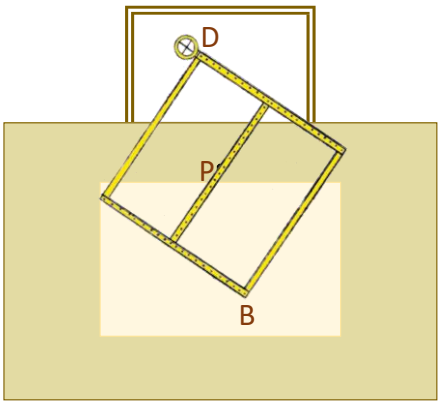
Take care during the operation that the feet "N" do not get stuck with the edges of the papers. Also, the pencil must be soft and sharp, having to put a little weight on it if it doesn't touch enough. Another recommendation from the author is that as freehand following a straight line can be difficult, you can mark only its beginning and its end, and then complete it with a ruler (this can be extrapolated to any layout, using the appropriate curved ruler and marking only the key points). In addition, when it is necessary to move from one area of the drawing to another, the pencil “M” must be raised so that it does not draw undesired lines.

Vertical copying (front views, perspectives)

Returning to the figure of the rhombus, now we place the pencil "M" in the vertex "B" and in "D" we place the sight "K", either substituting "AD" for another rod finished in the sight, or if the rod "AD" is actually made up of a long section and a short section, that one can be replaced by another equal short section, but with the "K" sight (always ensuring the final robustness of the "AD" section).

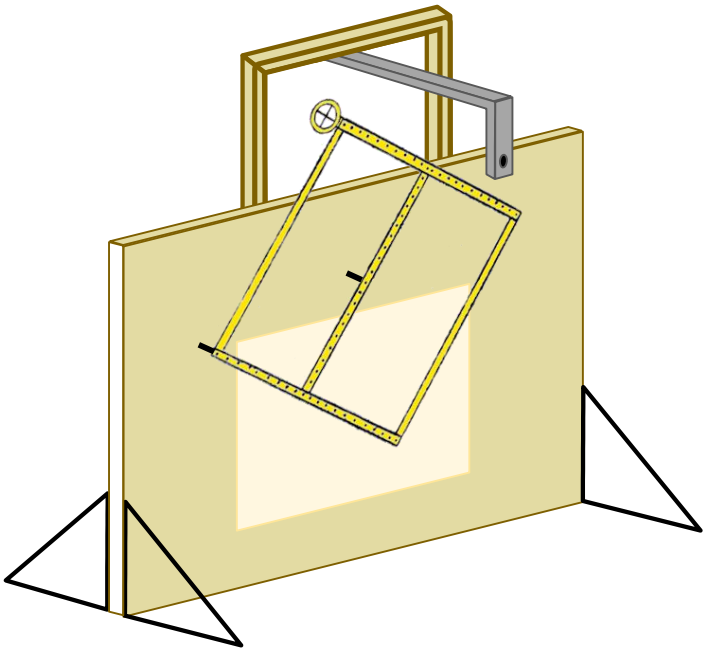






Our drawing (or copying) table is now vertical (ABDC) and on it we have an “AEFB” frame that will give the original framing. At point G the rhombus is fixed, that is, the foot “P” is placed. The arrangement of the rods is as explained before, depending on whether you only want to copy, to enlarge or to reduce.

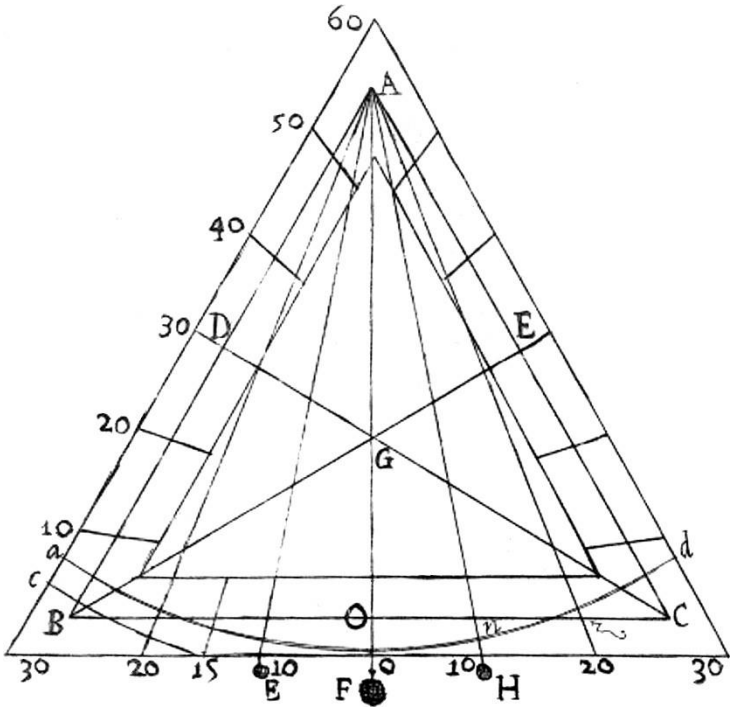
The last piece is a sight for the perspective point of view. It consists of an "L" shaped piece that hangs from the "H" point (fixed there with a screw). The assembly would be as follows:

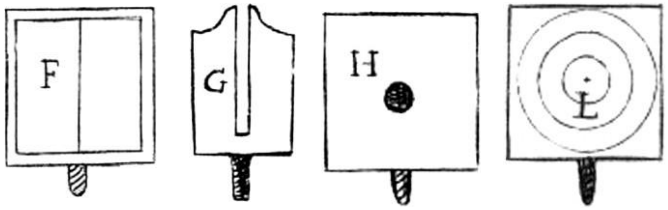


Looking from the hole "O" (which will be in the centre of the frame), we will move the sight "D" following the silhouette of the building or scene that we want "M", in "B", to draw on the paper.

The Equilateral Triangle

The triangle, that you can disassemble, is used to measure horizontal or vertical angles in the field. It consists of three brass plates four fingers wide (7.5 cm), three and a half feet long (105.7 cm), and "*the thickness of two Spanish dollars*" (about 6 mm [9], p.155). The plates fit together to form the triangle and the fixing screws (at each vertex and centred on the width of these) are perforated to insert the accessories: a wire or thread with a weight "F" (pendulum or plumb), and four sights (pinnules), whose use we will see now.

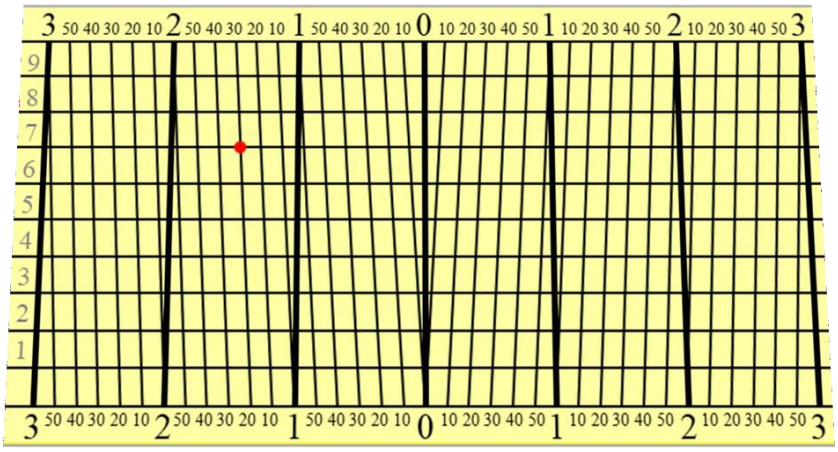


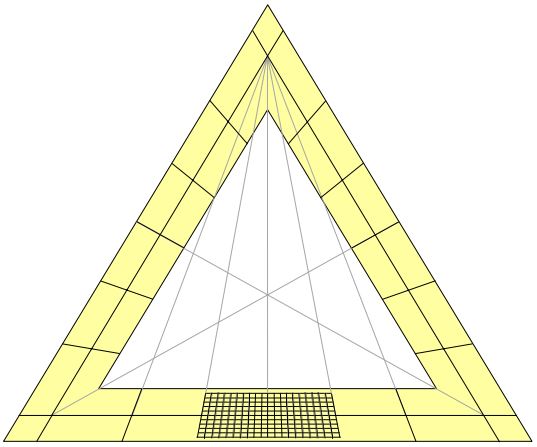


Drawing VII, figures 47 y 49

"A", "B" and "C" are the vertices of the equilateral triangle base for the engraved marks. “O” marks the centre between “B” and “C”. Between "B" and "O" there is 30º and dividing it into three parts (from the arc "aod") we will have the marks of 10º and 20º with respect to "O". In the same way we mark the 10º, 20º and 30º from “O” to “C”. And in the same way, the sides "AB" (with "D" as the centre) and "AC" (with "E" as the centre) are marked, although the numbering will go from 0º in "B" (or "C") to 60º (in "A").

Next, the degrees from 0º to 10º between “O” and “B” and between “O” and “C” are marked, and the minutes are also marked according to the following diagram (as in the brass rule):



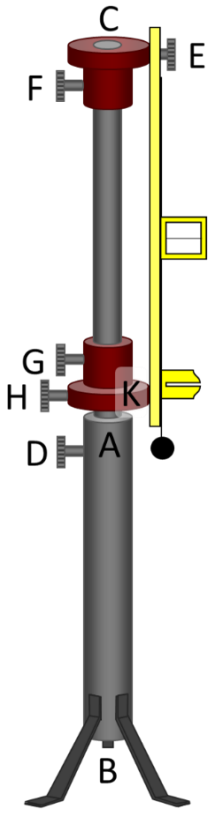


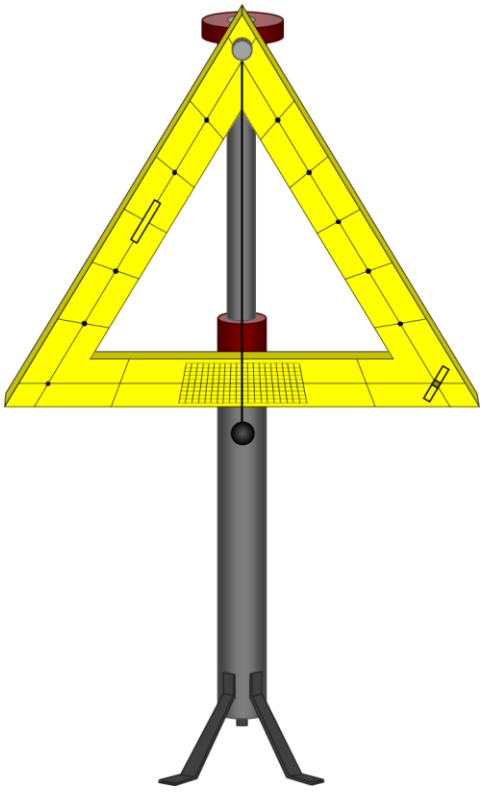
The horizontal lines divide the width of the plate into twelve parts, that is, 11 lines are drawn. The numbering of the degrees will be greater than that of the tens of minutes, to avoid confusion, (and the units of minutes, in grey in the image, are understood and are not marked). For example, the red dot marks the degree and 27 minutes “towards B” (“O” is 0º).

The “F” sight can be screwed into each of the points 10, 20, 30, 40, 50 or 60 both on the “AB” side and the “AC” side, and the “G” sight will be screwed on “C”, if “F” is at a point of “AB”, or at “B”, if “F” is at a point of “AC”. The pair “H” and “L” will replace the pair “F” and “G” when the observations are of the sun (we will see it later).

The stand of the triangle

The multi-purpose stand is also part of the chest, and it is used here to keep the triangle vertical, as in this simulation:





The stand is based in one cylinder inside another, both “*four quarters and a half*” high (distance “AB”, p.162). If the "quarter" is from a Castilian “vara”, this would be about 94 cm, with a total of 188 cm. The inner cylinder is slightly longer, to allow fixing with screw “D”. Three brackets screwed with two screws each form the tripod foot.

On this structure a complement is placed with the screws "F" to fix it and "E" to hang the triangle from its hole "x" that I have assumed, as it is not seen in the original drawing. The other complement, with the “G” screw and the “H” nail, is to ensure verticality. The triangle touches point "K" and the screw "G" is for fixing the complement. The vertical thread with the weight is thus flush with the triangle. The nail "H" can rotate on its axis, so that by putting another thread from one of the vertices "B" or "C" of the triangle and screwing it in "H", we can tighten it by turning "H", thus attracting the respective vertex, with which we can make the pendulum fall in the degree and minute marks towards “B” or towards “C”.

Apart from this layout, we can use the complement with screw "G" adding another three brackets screwed upwards (see figure 51, later); this will define a surface (in the form of a Y) on which we can place the triangle flat to measure horizontal

deviations. To do this, this complement must be placed on the upper part of the inner cylinder, and the brackets must have the size of "GO", "GD" and "GE" (see the use in the smaller triangle, later).

Instrument use

In the last figure you can see the sight "F" at 30º from the side "AB" and the sight "G" at the vertex "C". Looking from "G" we will try to align its slit with the index hairline in "F" and with the star we are observing. If the star is above, we can try putting "F" at 40º, and if it is below, at 20º, until we confirm it is less than 10º below or above the sight. Then, turning the nail "H" we will incline the triangle in the respective direction until we get the alignment of "G" with "F" and with the star.

If, for example, we have lowered “F” to 20º and then, pulling “B” while turning “H”, we have achieved alignment (it is over 20º but below 30º), then we see where the pendulum thread cuts the marks on the horizontal scale of degrees. If, for example, the crossing is at 1 degree and 27 minutes (like the red dot in the previous figure) then the star will be at 21 degrees and 27 minutes. Playing with the position of the sight "F" and "pulling" either the vertex "B" or "C" with the thread screwed in "H" we can read heights from 0º (the sight "F" at 10º and the pendulum at - 10º) up to 70º (the sight “F” at 60º and the pendulum at +10º).

To measure an angle greater than 70º, there are two alternatives. The first is to mark with degrees and minutes from “O” up to "B" and up to "C" (the entire side "BC") and proceed as before, "pulling" the vertices "B" or "C" with the thread wound on the nail “H”. The second is to put the “F” sight at point “A” and the “G” sight at point “z” or point “n”. The point "z" is at the intersection of the line "BC" with the 20º line from "O" to "C", which will allow us to measure angles from 70º to 80º. Point "n" is at the intersection of the line "BC" with the 10º line from "O" to "C", which will allow us to measure angles from 80º to 90º.

The first alternative (all “BC” in minutes) allows to measure with less movement of the sights (there is ± 30º of adjustment with the pendulum), and the second allows a first “coarse” adjustment with the sights and a second “fine adjustment” with the thread of “H” (and having fewer marks in “BC”).

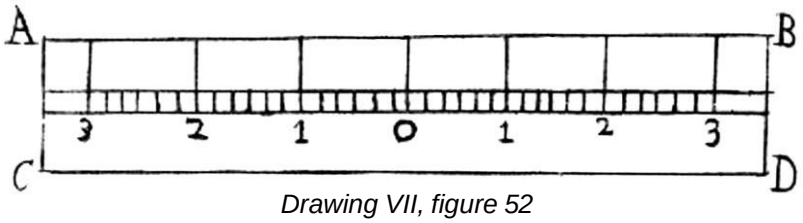
As I mentioned before, to measure the height of the sun, the “H” and “L” sights are used, since you do not look at the sun with your eye, but rather its brightness on “L” when going through the “H” hole. “H” replaces “F” and “L” replaces “G”. The “H” hole will create a light spot on “L” as we approach alignment, which we will achieve when the light spot is in the exact centre of “L”. At that moment the reading will be done on the pendulum as in the previous case. I imagine (the author does not say so) that with a piece of paper or something similar, you will look at where the point of light is initially, to know how to move the sight "H" or how to pull the vertices "B" or "C" (with the thread on the nail “H”), until the point is in the sight “L”.

Father Zaragoza does comment that the point of light can be larger or smaller depending on the apparent diameter of the sun (and therefore the concentric circles in "L"). And that in the hole of "H" you can put a concave crystal so that it focuses the light in a point on "L", although I think this will only work for a fixed distance between "H" and "L", and the author indicates with “H” in “A” (“*so that the rays unite at the distance of the side AC*”, p.168).

The other instruments

For the secondary instruments for topography, the author does not dedicate a chapter, but sections (for example, after the equilateral triangle).

The first is the **minor equilateral triangle**, which is only two and a half feet on a side (75.5 cm), “*and as wide as AC (Fig. 52)*” (p.169). Unfortunately, I have no way of measuring the "real" height of this figure:

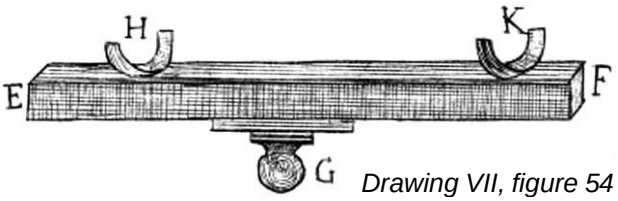


Drawing VII, figure 52

This triangle only has the numbered degree divisions plus the five of the tens of minutes. According to the author " *it gives more truth than astrolabes and semicircles and it is easier to handle*" (p.169). Placed horizontally on the tripod like the large triangle (each side screwed into an upper bracket), the wired triangle is placed on top of it, on the line "BC" (see the lines and letters of the major equilateral triangle), and with it locate the first reference (place, building...) by moving the equilateral triangle horizontally. Next, with the minor equilateral triangle still, the wired triangle is moved so that it points to the second reference, maintaining its crossing through vertex "C". The new position of the filar triangle will mark, on the smaller equilateral triangle, the angle between the two references (for example, if it ends according to "CD" the angle is 30º).

If the angle between references is greater than 60º, the wired triangle will be changed to "A" or "B" from crossing vertex "C" as appropriate, and the observer will stand on one or the other side of the smaller equilateral triangle, thus covering the 360º. The error due to this variation of the centre "*is insensitive at large distances*" (p.171).

The second is the **telescope**, made by the author and that can have two or four convex crystals, depending on whether you want to observe the firmament or something on earth (less clarity but not upside down images). Although it does not include an image of the telescope, it does describe the rod to support it, with two semicircles for positioning and to which it is tied with two ribbons (or strings...).

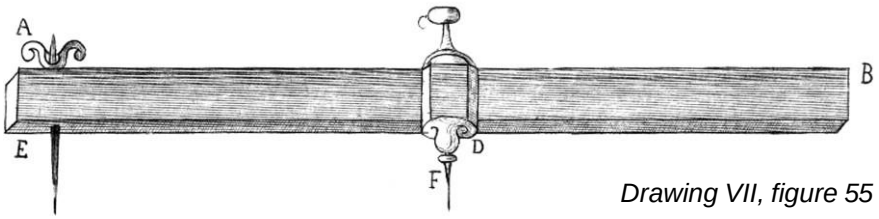


Drawing VII, figure 54

On the “G” knob it is attached to the multi-purpose stand so that the telescope can rotate in all directions and heights. In addition, the telescope has a disk with a reticule to draw objects at a distance (spots on the moon, an enemy square...).

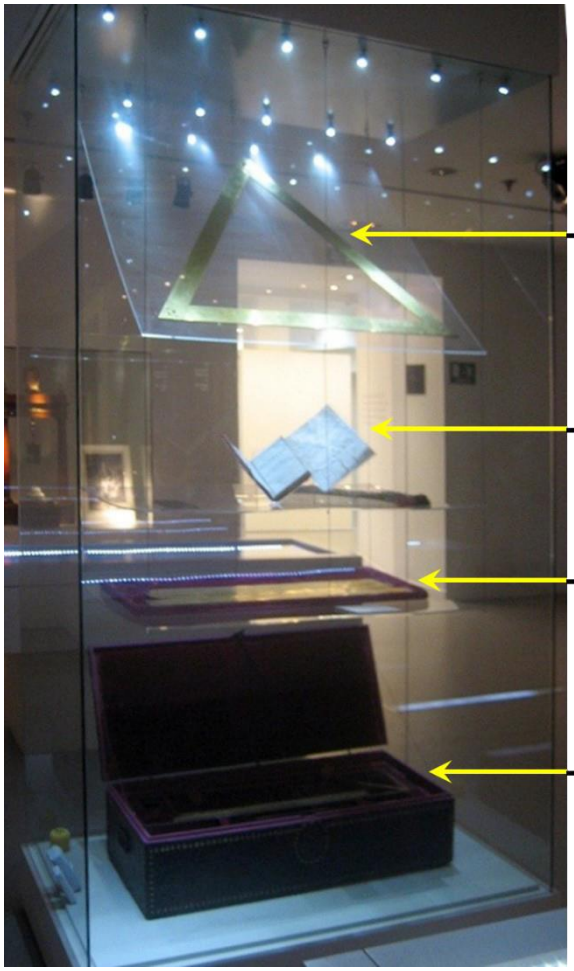
The third is the **chain** of 10 geometric steps (15.09 m) formed by elliptical links. Each foot (30.2 cm) is indicated with a small circular link, and the steps (every five feet) with a larger circular link.

The fourth is the **rod dividers**, which is used to take large measures of the sector (for example, to mark the frets on the guitar mentioned in the harmonic sector).



Drawing VII, figure 55

The last secondary instrument (I don't count the table or the multipurpose tripod stand) is the field **square** of one “vara” (83.5 cm) and of brass.



[7a]

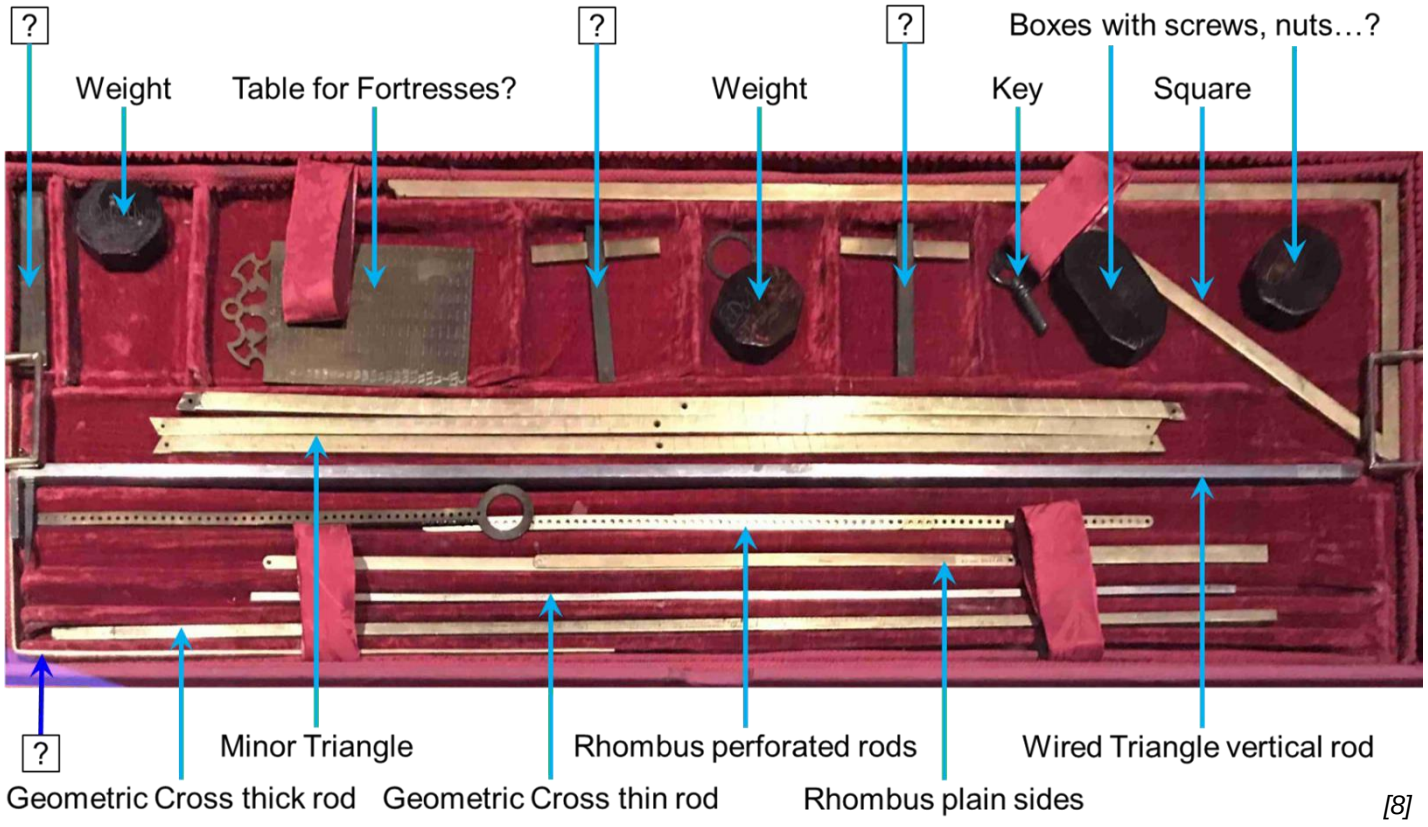
¿Where are the instruments?

In 2018 the chest could be seen in a thematic exhibition called "Cosmos" at the National Library of Madrid (BNM). As the chest "*appears in the inventories of the Royal Public Library (direct antecedent of the National) since the year of its foundation: 1711* [7]", I assume that it is still there, although I am not sure if it can be seen. In short, I have been lucky enough to find a few photos on the Internet, on the "Turismo Matemático" website [7], and our colleague Rafael Jené shared an image of said exhibition in the ARC forum [8]. From these I have tried to identify the instruments in the chest.

In the image on the left, from the 2011 exhibition, we can see one of the trays that includes the chest. I have been able to count four floors, although the last one, the bottom, is not clear. In this first tray I understand that there are the brass ruler and the sector, leaving space to also include the disassembled equilateral triangle.

Let me say that I haven't been able to identify all the instruments and components (not all of them are visible), and there is some items that I have not figured out what it is for. On the web [7] there is a comment from the museum itself in which it is recognized that the telescope already disappeared in the 19th century, I imagine that with its support rod.

The second level contains most of the instruments:



[8]

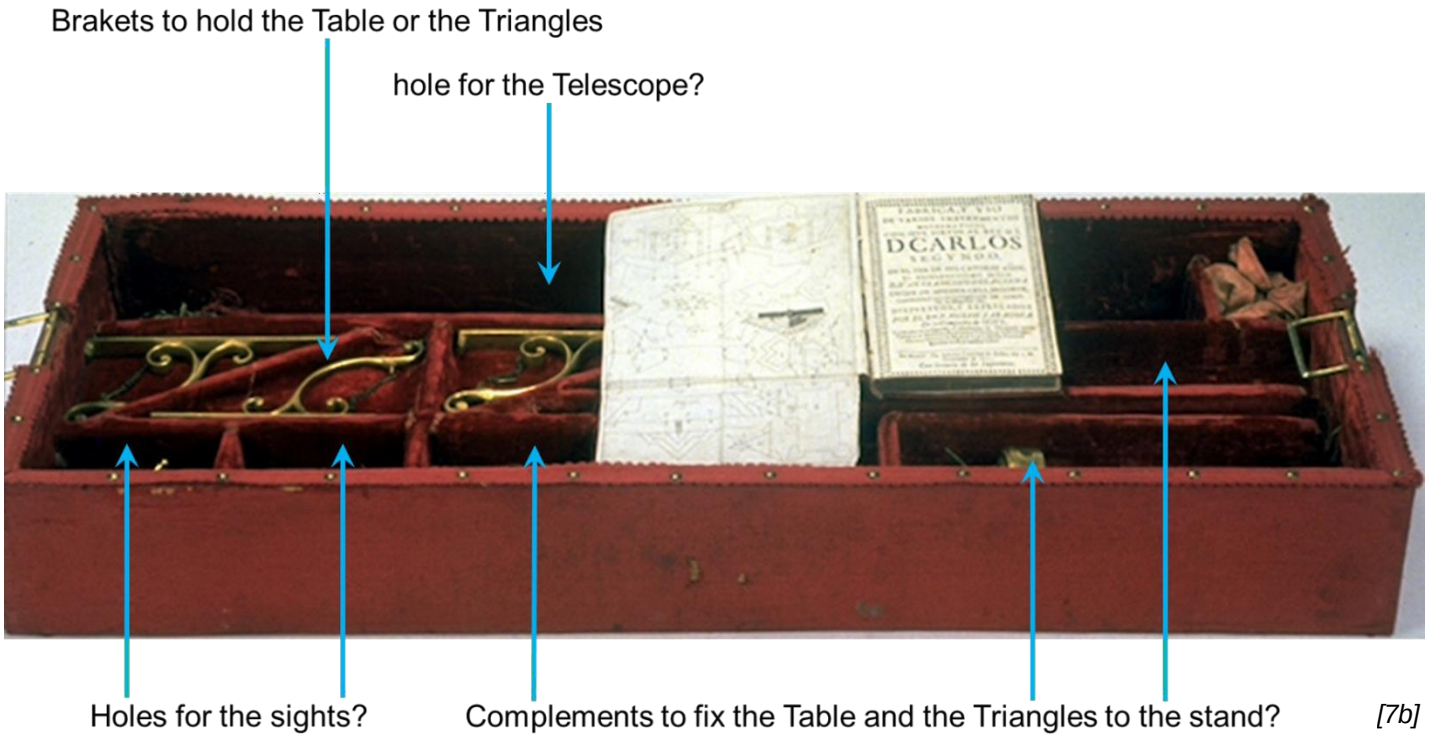
With a question mark I have marked those elements of which I do not know their use, or of which I am not sure, such as those that I assume are boxes of screws and nuts. I would like to mention what I suppose is a table for fortresses (explained

in the military sector), since by enlarging the image enough I intuit Roman numerals in the left column, from IV to XX, XXX, XL, LXX and C, although it seems to include (in addition to the extra rows) a column or two more than the book table (p.33):

Tabla de la Fortificacion Real

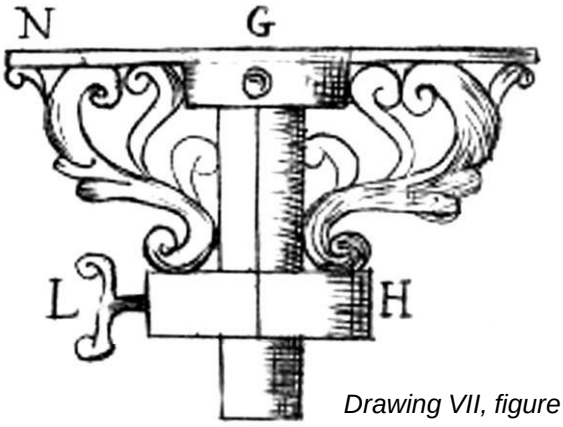
Figur.	Radio.	Lado.	Semigo la.	Tra- ues.	Angulo.
	Paſſos	Paßos.	Paſſos.	Paſſos	Grados.
IV	103.1	146.	23.	23.	90. 00
V	131.0	154.	27.	27.	72. 00
VI	160.0	160.	30.	30.	60. 00
VII	185.2	162	31.	30	51. 26.
VIII	214.1	164	32.	30	45. 00
IX	212 2	165	33.	30	40. 00
X	271.4	168	34.	30	36. 00
XI	301.3	170	35.	30	32. 44
XII	332.1	172	36.	30	30. 00
XV	418.2	174	37.	30	24. 00
XX	562.3	176	38.	30	18. 00
L	1414.1	178	39.	30	7. 12
Recta.	Infin.	180	40.	30	Recta.

On the third, deeper level, the bulky elements begin, and many of them cannot be seen:



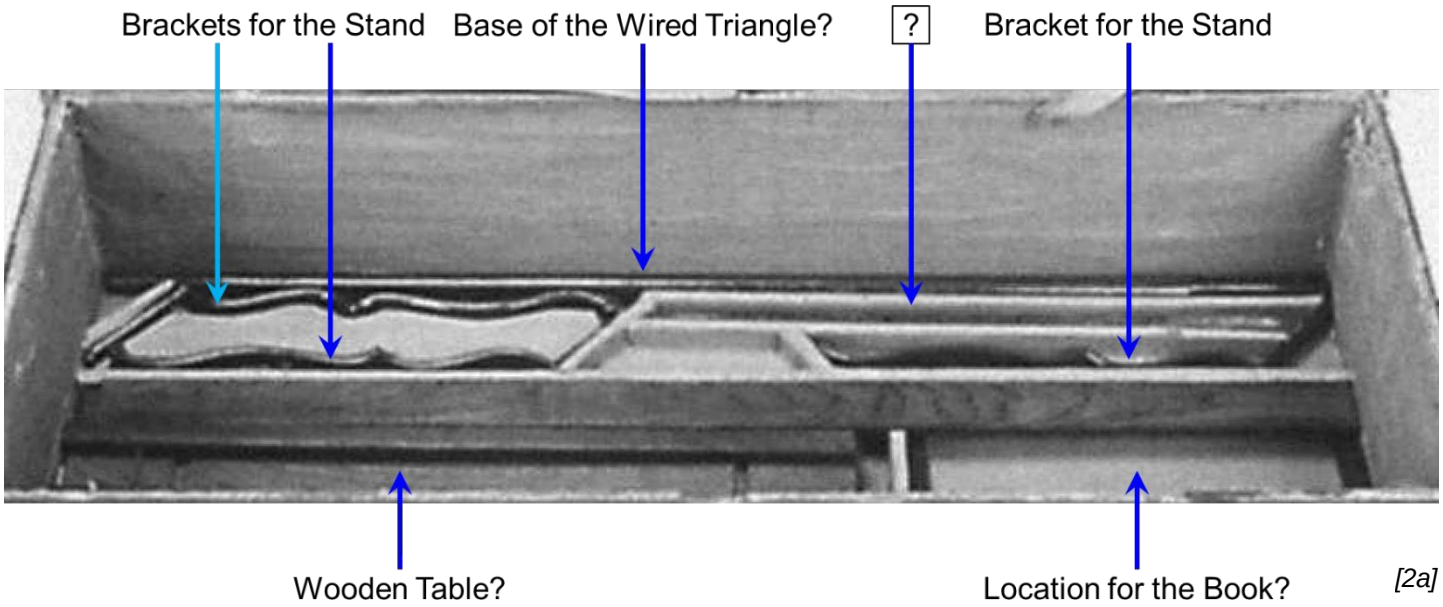
[7b]

Although the book appears in this image, I don't think this is where it is located. I recognize the brackets from their image in figure 51, in drawing VII, where they appear (“N”) already screwed to one of the stand complements (“GHL”), perhaps the only one that shows a bit in the photo (below to the right).



Drawing VII, figure 51

The fourth and last level is the bottom of the box, of which I only have a black and white image [2]:



[2a]

I guess that the pair of tubes that form the axis of the Multipurpose Stand should be at this level, perhaps over the accommodations of the Table and the Book. Neither have I found the 10-step Chain, the center of the Geometric Cross and the Rod Dividers, in addition to the sights, the threads and the special screws, although I suppose the latter are in the boxes on the second tray, and the Flemish Box, although I imagine that this went immediately to join the personal articles of Carlos II, being separated from the chest.

Final Comment

I hope I have presented in sufficient detail the instruments in this most interesting chest. As I have already said, the fact that it was a gift for a king leads me to believe that this proves the high interest, cost and technology for the time. But, after reading almost the entire book, I find the high mathematical and technological level of the explanations a bit shocking, considering that the addressee was a fourteen-year-old boy, of whom I have not read that he had outstanding intellectual capacity. Without going any further, the logarithms, which are explained in our educational system at the age of fifteen, are taken for granted and assumed in the book.

Perhaps the book had to be, in addition to a gift, a guide for engineers and mathematicians of the time. Father Zaragoza might have assumed, then, as the king's new teacher, that he would have time to explain everything that the disciple did not understand. In addition, the author's final comment is curious: "This is, My Lord, what in the brief space of a month the desire to serve Your Majesty has been able to achieve" (p.221-222). I find it hard to believe that it was possible to write the book and make all the instruments, starting from scratch, in a month. It seems more credible to me that he already had part of the instruments or that he commissioned them from other master craftsmen (the Jesuits Baltasar de Alcázar and Juan Carlos Andosilla collaborated, according to [3]), and that the book included chapters he had already written previously.

In any case, that does not take from his merit, as is detailed by [2] regarding the musical chapter, and regarding the fact that he was recognized as a supervisor of military projects (he also has a publication "Treatise on military architecture" from 1674). It is worth reading the bibliography of José Zaragoza, either from the RAH [3] or on Wikipedia itself.

Acknowledgments

It's funny how I got to know the author, the chest and his book. As I write this, Klaus Kühn already has his second version of “Calculating with Tones: the logarithmic logic of music” [6] ready, and in section 6.4.2 he makes a description of José Zaragoza's Harmonic Sector. To clarify some concepts in said book, since it is in old Castilian, not only in vocabulary and grammar, but also in spelling, Klaus contacted me. We had already had some previous contact related to José Zaragoza and Juan Caramuel, the other Jesuit who edited a table of logarithms at that time, the first Spaniard to do so, although not in Spain (which Zaragoza did).

So I must thank Klaus' tireless study, who came up with this document, who dared to read it, and who piqued my curiosity to undertake the writing that is now ending. A “maxi size” sector caught my attention and it only had two scales that coincided with those of the common designs of this instrument (either French or English version).

It is also curious that in [2] they establish the deep knowledge of these instruments in Spain, mentioning Michiel Coignet, (1549 - 1623), a Flemish mathematician at the Spanish court in Flanders, who might be a candidate for the invention of the sector, but it seems proven that between 1580-90 he systematically studied its potential, and this was the basis for the works of Galileo. It would be necessary to contrast his various bibliographical references with what Wikipedia says of Guidobaldo Del Monte (1545 - 1607), that "*his invention of a drawing instrument to construct regular polygons and divide a line into any number of segments was incorporated as another utility of the geometric and military sector of Galileo*”. Another research in need of an author...

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[1b] José Zaragoza: “Fábrica y uso de varios instrumentos matemáticos”. Vínculo en Internet

[2a] Cristina Bordas, Luis Robledo, Tess Knighton: “Jose Zaragoza's Box: Science and Music in Charles II's Spain”. Oxford Journals, Oxford University Press, Early Music, Vol. 26, No. 3 (Aug., 1998), pp. 391-413

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